Journal of Statistical Physics, Vol. 19, No. 5, 1978

Estimates of General Mayer Graphs. II. Long-Range Behavior of Graphs with Two Root Points Occurring in the Theory of Ionized Systems

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Received October 3, 1977; revised June 1, 1978

We find the asymptotic behavior of general Mayer 2-graphs (Mayer graphs with two root points), which occur in the theory of ionized systems. This problem arises when one wants to compute corrections to the Debye length for large values of the plasma parameter. For a given 2-graph $\Gamma(r)$ with Debye–Hückel lines e^{-r}/r, we prove the inequalities $C_m r^{-\lambda} e^{-\lambda r} \leqslant \Gamma(r) \leqslant \Gamma(r_0) C_M r^{3k-l} e^{-\lambda r}$, for any $r \geqslant r_0$, and where C_m and C_M are positive and finite constants which depend only on Γ. These bounds are finite whenever $\Gamma(r)$ is not infinite everywhere. The integers l, k, and λ denote, respectively, the number of lines of the graph Γ, its number of field points, and its local line connectivity (the maximum number of chains linking the root points, which have no line in common). From this result, we deduce that the simple irreducible 2-graphs dominant at large distances decay exponentially like e^{-r} and have an isthmus between the root points (an isthmus is a line whose deletion separates the graph into two disjoint components, each one containing a root point). We prove also that 2-graphs that have a number of lines $l > 3k + \lambda$ are infinite. We exhibit simple, irreducible prototypes satisfying this condition, for any $k \geqslant 6$. This implies that the Abe–Meeron theory of ionized gases as applied to a classical plasma is not free from divergences. Finally, we extend the preceding results to 2-graphs with lines $f_L = (e^{-r}/r)^{k_L}$, with k_L real positive. We prove that they still decay exponentially like $e^{-\lambda r}$, where λ is now the maximal flow in a network associated to Γ by assigning the capacity k_L to each line L.

KEY WORDS: Mayer graphs; Laplace integral; inequalities; local line connectivity; max-flow, min-cut theorem.

1. INTRODUCTION

The one-component plasma (OCP) consists of charged, classical point particles of a given sign in a uniform neutralizing background. The Debye

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length λ_D has a fundamental importance in the study of this system. It is defined as $-\lim_{r\to\infty}(1/r)\ln|h(r)|$, where $h(r)$ is the radial correlation function, and represents the distance at which a given charge begins to be screened by charges of opposite sign. For dilute plasmas {small values of the plasma parameter $\epsilon = [4\pi\rho(e^2/4\pi\epsilon_0 kT)^3]^{1/2}$}, the Debye length is equal to $\lambda_D{}^0 = (\epsilon_0 kT/\rho e^2)^{1/2}$, where ρ is the density of the system and T is its temperature. For dense plasmas (large values of ϵ), it has been shown[1] that

$$\lambda_D = \lambda_D{}^0[1 + \epsilon(\ln 3)/8 + \cdots]^{-1} \tag{1}$$

under the hypothesis that the long-range behavior of the potential of mean force $w(r)$ is given by the sum of chains made only of Debye–Hückel lines $b(r)$ and Abe–Meeron lines $B(r)$.[2,3] These are defined by the identities

$$b(r) = e^{-r}/r \tag{2}$$

$$B(r) = e^{-\epsilon b(r)} - 1 + \epsilon b(r) \tag{3}$$

To prove this hypothesis, a first important step consists in finding the behavior at large distances of 2-graphs (or Mayer graphs with two root points) with Debye–Hückel lines $b(r)$. This is the problem we investigate here.

In their important work, Del Rio and De Witt[1] have remarked that the particular 2-graphs made of κ chains in parallel (without *points* in common) decay exponentially like $e^{-\kappa r}$ [that is, one has $\lim_{r\to\infty}(1/r)\ln\Gamma(r) = -\kappa$]. Here, we prove that any given 2-graph decays exponentially like $e^{-\lambda r}$, where λ stands for the maximum number of chains linking the root points, which have no *line* in common.[2] This problem has also been investigated by Deutsch *et al.* for general 2-graphs.[7,8] But their proof is incorrect[3] and their method, even if it could be corrected, could only give upper bounds, for a

[2] These results were presented in detail in the course of a set of seminars given at the Laboratoire de physique et optique corpusculaire, Université Paris V.[4]

[3] Deutsch *et al.*[8] bound a given 2-graph by iterating the following algorithm. First, they choose a line, say (3, 4). Then they split the domain of integration into two parts, one where $r_{34} > r_0$, and the other where $r_{34} \leqslant r_0$, where r_0 is defined by the equality $f(r_0) = 1$. The integral in the first domain is bounded by the 2-graph obtained by deleting the line (3, 4), because one has $f_{34} < 1$. Then, they evaluate the integral in the second domain by assuming that $f_{3j} \sim f_{4j}$ for any j, when $r_{34} < r_0$. This enables them to replace f_{3j} by f_{4j} in the integral (this is their pinching procedure). But this is *not* possible, because the condition $r_{34} < r_0$ does not imply $f_{4j} \sim f_{3j}$ (one can have $f_{4j} = +\infty$ and $f_{3j} = 1$). This would be true only if one had, moreover, $r_{3j} \gg r_0$. So, still other terms ought to be taken into consideration, the decay of which is not known. Finally, nothing ensures that their pinching procedure gives, for a given *finite* 2-graph, "bounds" (that is, the quantities obtained from a given graph by the pinching procedure) that are finite. On the contrary, our upper bound (31) is finite whenever $\Gamma(r)$ is not infinite everywhere (i.e., is finite at least at one point).

given 2-graph, that decay exponentially like $e^{-\kappa r}$, where κ is the maximum number of chains without points in common.[4]

This paper is organized as follows. In Section 2, we give some definitions and notations, together with a useful homogeneity property of 2-graphs with Debye–Hückel lines e^{-r}/r. In Section 3, we rewrite a 2-graph in the form of an integral of Laplace's type. This enables us to obtain its asymptotic behavior simply by finding the minimum of the quantity $h_\Gamma = \sum_{L\in\mathscr{L}\Gamma} |\mathbf{R}_L|$ (which is the sum of the distances between pairs of points linked by a line of Γ). This minimum is computed in Section 4. We first prove that $\lambda(1, 2) \leqslant h_\Gamma$ for any configuration of the field points. Then, by making use of the max-flow, min-cut theorem, we exhibit configurations where h_Γ takes values arbitrarily close to $\lambda(1, 2)$. In Section 5, upper and lower bounds for $\Gamma(r)$ are deduced from the lower and upper bounds for h_Γ. In Section 6, we give sufficient conditions for a 2-graph with Debye–Hückel lines to be divergent. A first type of condition is obtained by requiring that the upper bound decays more rapidly at large distances than the lower bound. A second type is obtained by observing when the lower bound can become infinite. In Section 7, we generalize the preceding results to 2-graphs with lines $(e^{-r}/r)^\alpha$, with α real, positive.

2. DEFINITIONS AND NOTATIONS

2.1. Definition of a 2-Graph

A 2-graph[5] is a multiple integral of the following type[9,10]:

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2) = \int_{\Lambda_\infty{}^k} \prod_{L\in\mathscr{L}\Gamma} f_L(\mathbf{r}_i, \mathbf{r}_j)\, d\mathbf{r}_3 \cdots d\mathbf{r}_{k+2} \tag{4}$$

[4] Deutsch *et al.*[8] call this quantity the degree of convection I of the graph. In Ref. 8, they corrected their initial results[7] to recover our $e^{-\lambda r}$ decay. Their argument is based on the assertion that $\lambda = \kappa$ for bridge graphs (i.e., simple, irreducible graphs). This is nevertheless false, as can be seen in Fig. 3c. The two quantities κ and λ are generally different, in the same way that the number of points and lines in a graph are generally different (although one can clearly have graphs with the same number of points and lines, and similarly graphs with the same local point connectivity κ and local line connectivity λ). In fact, only the inequality $\kappa \leqslant \lambda$ holds true. To see that their method cannot give upper bounds decaying like $e^{-\lambda r}$, it is sufficient to find a graph where *all* lines belong to *all* maximal sets of line-disjoint chains. So, the deletion of *any* line gives a graph with a local line connectivity equal to $\lambda - 1$. The graph of Fig. 3c has this property. We prove in this paper that it decays exponentially like e^{-3r}, because $\lambda = 3$. But the pinching procedure of Deutsch *et al.* gives an upper bound which is a sum of graphs, where at least one of them has one line less. So, even if their bounding procedure could be corrected, their bound could not decay faster than e^{-2r} anyway.

[5] This is usually called a graph with two root points, or a two-rooted graph. In our work, we must make a clear distinction between a graph and the multiple integral it

where the symbols have the same meaning as in the first paper of this series.[10] We recall them briefly. Γ is a graph with two root points, k field points, and l lines L joining the points i and j. The set of lines of Γ is denoted by $\mathscr{L}\Gamma$. In (4), the product runs over all lines of $\mathscr{L}\Gamma$, and the integration runs over the k field points varying in the infinite domain Λ_∞.

2.2. Definition of the f_L and Notations of a 2-Graph

The problem that is investigated in this paper is to find, for large distances r_{12} between the root points, the asymptotic behavior of any 2-graph whose lines f_L are powers of Debye–Hückel lines. This means that the f_L are defined by the identity

$$f_L(\mathbf{r}_i, \mathbf{r}_j) = (e^{-r_L}/r_L)^{k_L} \tag{5}$$

where r_L denotes the distance between particles i and j:

$$r_L = |\mathbf{r}_i - \mathbf{r}_j| \tag{6}$$

Sometimes we will write r_{ij} instead of r_L. Note that the lines $f_L(\mathbf{r}_i, \mathbf{r}_j)$ are translation invariant, and so $\Gamma(\mathbf{r}_1, \mathbf{r}_2)$ has the same property. This will usually be written

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2) = \Gamma(r_{12}) \tag{7}$$

If Γ is represented according to Mayer's convention,[11] we have $k_L = 1$ for any L, and Γ is a multigraph, i.e., it has k_{ij} lines joining points i and j.

If Γ is represented according to our convention,[9,10] it has a single line $L = (i, j)$ joining points i and j. Moreover, to each line L is assigned the number k_L, which is called the capacity of the line (k_L is equal to the k_{ij} of Mayer's representation). In the graphical representation of Γ, k_L is written near the line L.[9,10]

To illustrate this, the 2-graph $\gamma(r_{12})$, defined by

$$\gamma(r_{12}) = \int_{\Lambda_\infty} (e^{-r_{13}}/r_{13})^2 e^{-r_{32}}/r_{32}\, d\mathbf{r}_3 \tag{8}$$

has been represented in the two different ways in Fig. 1. In Sections 1–5, the k_L will take only positive integer values. Although the two types of con-

represents, and at the same time indicate explicitly how many root points or variables there are. We have chosen the names n-rooted graph for the graph-theoretical concept, and n-graph for its associated integral, because they satisfy both conditions and are sufficiently simple. Similarly, a graph is denoted by Γ and its associated integral by $\Gamma(\mathbf{r}_1, \ldots, \mathbf{r}_n)$. Note that a 1-graph is then denoted by $\Gamma(\mathbf{r}_1)$, although it is independent of $\mathbf{r}_1$.

Fig. 1. The 2-graph $\gamma(r_{12})$ defined by Eq. 8 is represented here according to: (a) the usual Mayer convention[11]; (b) our convention.[9,10] This latter one enables us to represent 2-graphs whose integrand is a product of functions $f^{\alpha}(\mathbf{r}_i, \mathbf{r}_j)$, for any *real* positive α, while the Mayer convention can be used only if α is an integer.

ventions can be used, we will represent 2-graphs according to Mayer's convention, because it involves more intuitive concepts and reasoning than ours. But in Section 7, we will consider 2-graphs where some k_L can take noninteger values.[6] So, 2-graphs will then be represented according to our convention, because it is the only one possible in this case.

2.3. Definition of the 2-Graphs Studied

Insofar as we study each 2-graph individually, we can restrict ourselves to those that are simple irreducible. This is because any 2-graph can be factorized into a product of 1-graphs and simple, irreducible 2-graphs by the usual theorems of integration in coordinate space.[10,12] Simple, irreducible[7] 2-graphs occur in the development of the potential of mean force $w(r)$ in powers of the density ρ.[12] Some examples are given in Fig. 2. Finally, we could also restrict ourselves to simple irreducible 2-prototypes[11] (a 2-prototype is a 2-graph where each field point has a degree equal at least to three). But it is unnecessary to impose this restriction before Section 6.

[6] In the Mayer developments of the distribution functions, there are only 2-graphs with integer k_L. Nevertheless, we are also interested in 2-graphs with noninteger k_L, because we will need upper bounds of such 2-graphs to obtain improvements of (31) and (32). This will be the subject of a subsequent paper.

[7] A two-rooted graph is irreducible if each field point belongs to a chain of field points linking the root points. An irreducible two-rooted graph is simple if each pair of field points is linked by a chain of field points.[12] A 2-graph is irreducible if its associated two-rooted graph is irreducible.

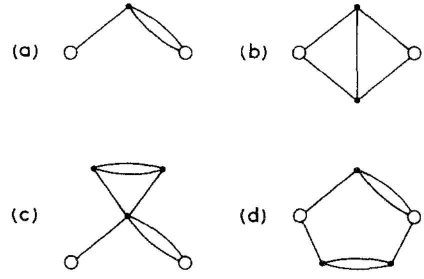


Fig. 2. The 2-graphs (a) and (b) are simple, irreducible. (c) is reducible because it has an articulation point, and (d) is irreducible but not simple because the root points form an articulation set.

2.4. A Useful Homogeneity Relation

Let us suppose for now that we have taken in (4) the lines $f_L(\mathbf{r}_i, \mathbf{r}_j) = e^{-\alpha r_{ij}}/r_{ij}$ instead of (5). Let us then denote the corresponding 2-graph, defined in (4), by $\Gamma(r_{12}; \alpha)$ to indicate explicitly that it depends on α. This enables us to write down the following homogeneity property[1]:

Lemma 2.1. One has the identity (provided the integral converges)

$$\Gamma(r_{12}; \alpha) = \alpha^{l-3k}\Gamma(\alpha r_{12}; 1) \tag{9}$$

Here, l denotes the number of lines of Γ in the Mayer representation:

$$l = |\mathscr{L}\Gamma| = \sum_{L\in\mathscr{L}\Gamma} k_L \tag{10}$$

Note that this is also the total capacity of the graph, in our representation, defined as the sum of the capacities of all the lines of Γ, and where the capacity of a line $e^{-\alpha r}/r$ is defined to be α.

The relation (9) is obtained immediately by making the change of variables $\mathbf{r}_i = \alpha^{-1}\mathbf{R}_i$. A consequence of (9) is that there would be no gain of generality in studying the decay of 2-graphs with lines $e^{-\alpha r}/r$.

3. Reformulation of $\Gamma(r_{12})$

To find the exponential decay of $\Gamma(r_{12})$, it is particularly convenient to express it in the form of an integral of Laplace's type.[13] To this end, let us regroup the exponentials together and set

$$\mathbf{r}_i = r_{12}\mathbf{R}_i \tag{11}$$

With this change of variables, $\Gamma(r_{12})$ becomes, by the same homogeneity considerations as before

$$\Gamma(r_{12}) = r_{12}^{3k-l}\int_{\Lambda_\infty{}^k} \exp\Big(-r_{12}\sum_{L\in\mathscr{L}\Gamma}|\mathbf{R}_L|\Big)\prod_{L\in\mathscr{L}\Gamma}\frac{1}{|\mathbf{R}_L|}\,d\mathbf{R}_3\cdots d\mathbf{R}_{k+2} \tag{12}$$

We have set, as in (6),

$$\mathbf{R}_L = \mathbf{R}_i - \mathbf{R}_j \tag{13}$$

where i and j denote the ends of line L. Notice that, by (11), the distance between the root points 1 and 2 is now

$$|\mathbf{R}_{12}| = R_{12} = 1 \tag{14}$$

We have written explicitly the absolute values in (12) in order to make further developments clearer.

A useful result about $\Gamma(r_{12})$, which can be obtained straightforwardly from its new form (12), is the following:

Lemma 3.1. If $\Gamma(r_{12})$ is not infinite everywhere (i.e., if there exists a certain r_0 such that $\Gamma(r_0) < +\infty$), then $\Gamma(r_{12})$ exists[8] for any $r_{12} \geqslant r_0$ and $r_{12}^{l-3k}\Gamma(r_{12})$ is monotonically decreasing for any $r_{12} \geqslant r_0$.

This is a consequence of the Lebesgue dominated convergence theorem.[14]

Remark 1. Looking for upper bounds for a given 2-graph $\Gamma(r_{12})$ has a meaning only if the 2-graph itself is not infinite everywhere. So, we will suppose from now on that there exists an r_0 such that $\Gamma(r_0) < +\infty$. Therefore, the preceding lemma ensures that the problem we are interested in, namely looking for the asymptotic decay of $\Gamma(r_{12})$, has a meaning, too, whenever such an r_0 exists, because $\Gamma(r_{12})$ is finite for any r_{12} sufficiently large.

Remark 2. The question of knowing whether or not there exists such a point r_0 will be considered in more detail in Section 6. Let us simply note here that the set of 2-graphs not infinite everywhere is clearly not void. For, it contains the set of 2-chains with any number of field points, and these 2-chains are known to be finite.[1]

3.1. Description of the Laplace Method

An integral of Laplace's type reads, in one dimension,[13]

$$\Gamma(r) = \int_a^b e^{-rh(t)} g(t)\, dt \tag{15}$$

and thus (12) indeed has a form similar to (15).

The crucial point in the Laplace method is that for large values of r, only the neighborhood of t_0 contributes to $\Gamma(r)$, where t_0 is the point where $h(t)$ reaches its minimum. Therefore, one has the asymptotic equivalence[13]

$$\Gamma(r) \sim e^{-rh(t_0)} [2\pi/r\,|h''(t_0)|]^{1/2} g(t_0) \tag{16}$$

under the following conditions:

(i) $h(t)$ is twice differentiable in (a, b).
(ii) $h'(t_0) = 0$, $h''(t_0) \neq 0$, $a < t_0 < b$.
(iii) $h(t) > h(t_0)$, $\forall t \neq t_0$ (t_0 is an absolute minimum).
(iv) $g(t)$ is continuous in (a, b).

[8] We have not succeeded in proving that, if $\Gamma(r_{12})$ exists for a given value r_0, it exists for all $r_{12} > 0$. By Lemma 3.1, we would just need to prove that $\Gamma(\epsilon; 1)$ exists if $\Gamma(r_0; 1)$ exists, for $\epsilon < r_0$. This seems to be true, because divergences can come only from small distances R_{ij}, and for such distances, we have $\exp(-r_0 R_{ij}) \sim 1 \sim \exp(-\epsilon R_{ij})$. Note that the homogeneity property (9) is of no help, because R_{12} is a fixed parameter in our problem. Here, we need only the weaker result given in Lemma 3.1, because we are interested only in the asymptotic behavior of $\Gamma(r_{12})$.

We will say that the principal (or exponential) decay of $\Gamma(r)$ is in $e^{-rh(t_0)}$, and that its complementary (or power) decay is in $r^{-1/2}$.

For multidimensional integrals, one has a formula quite analogous to (16), which gives both the principal and the complementary decay, if $h(t)$ has an absolute minimum at a point t_0 (t is now a multidimensional variable),(15) and if the Hessian (15) of $h(t)$ is nonnull at t_0. Note that, for a different set of conditions, one can have formulas that differ from (16) by a power of r,(13) but the important point for us is that the factor $e^{-rh(t_0)}$, that is, the principal decay of $\Gamma(r)$, remains unchanged.

3.2. Application to 2-Graphs with Debye–Hückel Lines

The preceding considerations suggest that the principal decay of $\Gamma(r_{12})$ ought to be determined by the minimum of the quantity

$$h_\Gamma(\mathbf{R}_3,\ldots,\mathbf{R}_{k+2}) = \sum_{L\in\mathscr{L}\Gamma} |\mathbf{R}_L| \tag{17}$$

and this is what we are going to prove now. Unfortunately, it will be seen below that the minimum is *not*, in general, reached at a point, but on a *set* of points, so that it is not possible to get the complementary decay (see Section 3.1) of $\Gamma(r_{12})$ with the usual formulas. To find this complementary decay, it would be necessary to find the set of points where the minimum of h_Γ is reached. This is a more difficult problem and it will be investigated in a subsequent paper. Here, we look only for the principal decay of $\Gamma(r_{12})$. To solve this problem, we need only to find the value μ of the minimum of h_Γ.

4. COMPUTATION OF THE MINIMUM μ of h_Γ

Let us first note that $h_\Gamma(\mathbf{R}_3,\ldots,\mathbf{R}_{k+2})$ is a finite sum of nonnegative continuous functions, and so its minimum μ does exist, is nonnegative, and is reached on a certain set of points $\Delta \subset \Lambda_\infty{}^k$. Having seen that μ exists, we want now to prove:

Theorem 4.1. The minimum μ of $h_\Gamma(\mathbf{R}_3,\ldots,\mathbf{R}_{k+2})$ is equal to the local line connectivity $\lambda(1, 2)$ of Γ.

$\lambda(1, 2)$ is defined in the following way:

Definition 4.1. The local line connectivity $\lambda(1, 2)$ is equal to the maximum number of line-disjoint chains[9] (chains without lines in common) linking the root points 1 and 2.

[9] A chain is a sequence of lines of the form $(i_1, i_2), (i_2, i_3),\ldots, (i_{k-1}, i_k)$, where all points are distinct.

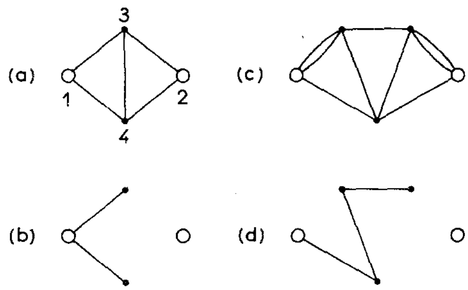


Fig. 3. Illustration of the local line connectivity $\lambda(1, 2)$. In the 2-graph (a) one has two line-disjoint chains, one made of the lines (1, 3) and (3, 2), and the other made of the lines (1, 4) and (4, 2). (b) represents a line-cutset of (a). It has two lines and so, by the max-flow min-cut theorem, the local line connectivity of (a) is equal to 2. In the 2-graph (c) one can find three line-disjoint chains, and the line-cutset (d) has three lines. Therefore, for (c) we have $\lambda(1, 2) = 3$.

This is illustrated in Figs. 3a and 3b.

By virtue of the max-flow, min-cut theorem,[16,17] the maximal number of line-disjoint chains between 1 and 2 is equal to the minimal number of lines that separate[10] the root points. Therefore, for the simplest 2-graphs, $\lambda(1, 2)$ can be computed very easily. For, suppose we have found a set of m line-disjoint chains; if we can find also a line-cutset with m lines, the max-flow, min-cut theorem ensures that $m = \lambda(1, 2)$. For example, Figs. 3a and 3b show that $\lambda(1, 2) = 2$, and Figs. 3c and 3d give $\lambda(1, 2) = 3$. In the general case, there are known algorithms to compute $\lambda(1, 2)$.[16,17]

4.1. Lower Bound for μ

We have restricted our study to simple, irreducible 2-graphs, and therefore there is at least one chain going from point 1 to point 2. For a given chain C, it is possible to write

$$\mathbf{R}_{12} = \sum_{L \in \mathscr{L}C} \mathbf{R}_L \tag{18}$$

where $\mathbf{R}_L$ was defined in (13) to be

$$\mathbf{R}_L = \mathbf{R}_i - \mathbf{R}_j, \qquad L = (i, j)$$

But, from (14), we have $R_{12} = 1$. So, by applying the triangular inequality to (18), we obtain, for any chain C linking the root points,

[10] A set of lines is said to separate the points 1 and 2 if their deletion gives two connected components, one of them containing the point 1 and the other the point 2.[18] A line-cutset is a set of lines that separate the root points. This latter definition is illustrated in Figs. 4c and 4d.

$$1 \leqslant \sum_{L \in \mathscr{L}C} |\mathbf{R}_L| \tag{19}$$

Let us now choose a maximal set of line-disjoint chains C_k. We have $\mathscr{L}C_m \cap \mathscr{L}C_n = \varnothing$ for any pair of integers m and n, $m \neq n$. This enables us to write

$$\sum_{m=1}^{\lambda(1,2)} \sum_{L \in \mathscr{L}C_m} |\mathbf{R}_L| = \sum_{L \in \bigcup_m \mathscr{L}C_m} |\mathbf{R}_L| \tag{20}$$

But (19) shows that the lhs of (20) is bounded below by $\lambda(1, 2)$ and, because $\bigcup_m \mathscr{L}C_m \subset \mathscr{L}\Gamma$, the rhs of (20) is bounded above by $h_\Gamma(\mathbf{R}_3,..., \mathbf{R}_{k+2})$. This gives the desired inequality:

$$\lambda(1, 2) \leqslant \mu \tag{21}$$

4.2. Upper Bound for μ

To prove the converse inequality $\lambda(1, 2) \geqslant \mu$, we need to use the max-flow, min-cut theorem[16,17] (or Ford–Fulkerson theorem). As we already said, it tells us that the maximal number of line-disjoint chains between 1 and 2 is equal to the minimum number of lines that separate 1 and 2.

Let then $\mathscr{C} = \{L_1, L_2,..., L_{\lambda(1,2)}\}$ be a family of such lines. For a given simple, irreducible 2-graph, this family is not void, because $\lambda(1, 2) \geqslant 1$. We have then $\Gamma - C = \Gamma_1 \cup \Gamma_2$, with Γ_1 and Γ_2 disjoint (i.e., $\Gamma_1 \cap \Gamma_2 = \varnothing$). Moreover, one end of line L_i belongs to Γ_1, while the other belongs to Γ_2. The line-cutset will be displayed by symbolizing Γ as shown in Fig. 4.

Let us now call $i_1, j_1,...$ the points of Γ_1, and $i_2, j_2,...$ the points of Γ_2. We can realize the abstract graph Γ in the usual space $\mathbb{R}^3$, by regrouping all the points of Γ_1 in a sphere S_1 centered at point 1 and with radius η, and all the points of Γ_2 in a sphere S_2 centered at point 2, with the same radius η. This is illustrated in Fig. 5. We have then

$$|\mathbf{R}_{i_1} - \mathbf{R}_1| < \eta \qquad \forall i_1 \in \mathscr{P}\Gamma_1 \tag{22}$$

$$|\mathbf{R}_{i_2} - \mathbf{R}_2| < \eta \qquad \forall i_2 \in \mathscr{P}\Gamma_2 \tag{23}$$

where $\mathscr{P}\Gamma$ stands for the set of points of Γ. This gives immediately, by applying the triangular inequality

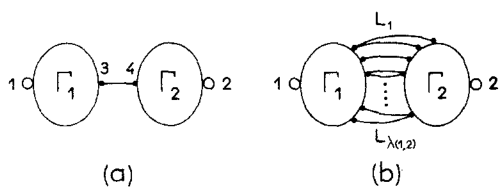


Fig. 4. General structure of 2-graphs with a given local line connectivity. (a) $\lambda(1, 2) = 1$. In this case we have an isthmus: the line (3, 4). (b) $\lambda(1, 2) = n$.

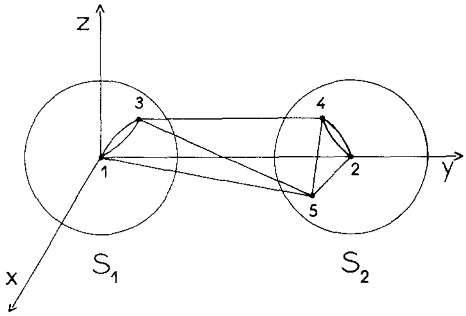


Fig. 5. A realization, in the real space $\mathbb{R}^3$ of the 2-graph (c) of Fig. 3.

$$|\mathbf{R}_{i_1} - \mathbf{R}_{j_1}| < 2\eta \qquad \forall i_1, j_1 \in \mathscr{P}\Gamma_1 \tag{24}$$

$$|\mathbf{R}_{i_2} - \mathbf{R}_{j_2}| < 2\eta \qquad \forall i_2, j_2 \in \mathscr{P}\Gamma_2 \tag{25}$$

$$|(\mathbf{R}_{i_1} - \mathbf{R}_{j_2}) - \mathbf{R}_{12}| < 2\eta \qquad \forall i_1 \in \mathscr{P}\Gamma_1, \quad \forall j_2 \in \mathscr{P}\Gamma_2 \tag{26}$$

These inequalities are valid for any pair of points, and thus a fortiori for all the lines of Γ.

As the subgraphs Γ_1, Γ_2, and $\mathscr{C}$ have no line in common, and as their union gives back Γ, we can split h_Γ into three parts:

$$h_\Gamma = \sum_{L \in \mathscr{L}\Gamma_1} |\mathbf{R}_{i_1} - \mathbf{R}_{j_1}| + \sum_{L \in \mathscr{L}\Gamma_2} |\mathbf{R}_{i_2} - \mathbf{R}_{j_2}| + \sum_{L \in \mathscr{L}\mathscr{C}} |\mathbf{R}_{i_1} - \mathbf{R}_{j_2}| \tag{27}$$

From the preceding inequalities, we see that the first two sums can be made negligible provided η is sufficiently small, and the last one is approximately equal to $\lambda(1, 2)|\mathbf{R}_{12}| = \lambda(1, 2)$. More precisely, we have

$$|h_\Gamma - \lambda(1, 2)|\mathbf{R}_{12}|| \leqslant 2\eta(l_1 + l_2) + \sum_{L \in \mathscr{L}\mathscr{C}} ||\mathbf{R}_{i_1} - \mathbf{R}_{j_2}| - |\mathbf{R}_{12}|| \tag{28}$$

where l_1 and l_2 stand, respectively, for the numbers of lines of Γ_1 and Γ_2, and where the lines of $\mathscr{C}$ are denoted by (i_1, j_2). This gives

$$|h_\Gamma - \lambda(1, 2)| \leqslant 2l\eta \tag{29}$$

because of the identity $l_1 + l_2 + \lambda(1, 2) = l$.

We have thus exhibited a set of points $\mathbf{R}_3, \ldots, \mathbf{R}_{k+2}$, such that h_Γ is arbitrarily close to $\lambda(1, 2)$. This proves that $\mu \leqslant \lambda(1, 2)$, and completes the proof of Theorem 4.1.

We can see now that the minimum of h_Γ is not, in general, reached at a point, but on a set of points. It is sufficient to consider the simplest irreducible 2-graph, that is, a chain with two lines. We have then

$$h_\Gamma(\mathbf{R}_3) = |\mathbf{R}_{13}| + |\mathbf{R}_{32}| \tag{30}$$

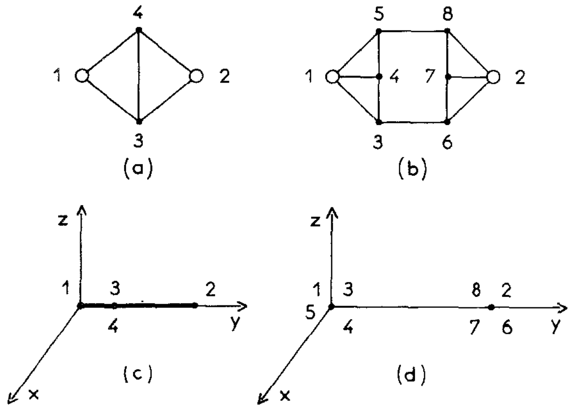


Fig. 6. The minimum of h_Γ is reached on a set of points for the 2-graph (a) and at a single point for (b). The realizations in the real space of the 2-graphs (a) and (b) when h_Γ is minimum are given respectively in (c) and (d).

and the minimum $|\mathbf{R}_{12}|$ of h_Γ is reached when $\mathbf{R}_3$ belongs to the segment of the line that joins the root points (in $\mathbb{R}^3$). For the 2-graph of Fig. 6a, the minimum is reached when $\mathbf{R}_3$ and $\mathbf{R}_4$ come to the same point on the preceding segment. Note nevertheless that μ can be reached sometimes at only one point, as in Fig. 1a or in Fig. 6b. In the latter case, h_Γ reaches its minimum at the point $(\mathbf{R}_1, \mathbf{R}_1, \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_2, \mathbf{R}_2)$.

5. ASYMPTOTIC BEHAVIOR OF A GENERAL 2-GRAPH WITH DEBYE–HÜCKEL LINES

The main theorem we will prove now is the following:

Theorem 5.1. A given irreducible 2-graph $\Gamma(r_{12})$ with l lines e^{-r}/r, k field points, and a local line connectivity equal to λ, has the following upper and lower bounds:

$$\Gamma(r_{12}) < \Gamma(r_0)C_M(r_0)r_{12}^{3k-l}e^{-\lambda r_{12}}, \qquad \forall r_{12} > r_0 \tag{31}$$

$$C_m(r_{12})r_{12}^{-\lambda}e^{-\lambda r_{12}} < \Gamma(r_{12}), \qquad \forall r_{12} > 0 \tag{32}$$

which are finite whenever $\Gamma(r_{12})$ is not infinite everywhere. The $C_M(r_0)$ and $C_m(r_{12})$ are the finite quantities

$$C_M(r_0) = r_0^{l-3k}e^{\lambda r_0} \tag{33}$$

$$C_m(r_{12}) = C_1(1, 1)C_2(1, 1)e^{-2\lambda}(1 + 2r_{12}^{-1})^{-\lambda} \tag{34}$$

$C_1(1, 1)$ and $C_2(1, 1)$ are positive constants defined in Section 5.2, and r_0 is an arbitrary, real, positive number, chosen such that $\Gamma(r_0) < +\infty$.

Remark. We have obtained only upper bounds, but not the exact asymptotic decay, because the minimum of h_Γ is not reached on a unique point, but on a set of points, as we already noted.

Before proving Theorem 5.1, let us first draw some consequences of this theorem:

Corollary 5.1. An irreducible 2-graph $\Gamma(r_{12})$ with Debye–Hückel lines e^{-r}/r decreases exponentially like $\exp(-\lambda r_{12})$, where λ is the local line connectivity of Γ, provided $\Gamma(r_{12})$ is not infinite everywhere. Conversely, a 2-graph that decreases exponentially like $\exp(-nr_{12})$, with n an integer, is made of two disjoint connected graphs Γ_1 and Γ_2 linked by n lines, and which contain, respectively, the root-points 1 and 2 (see Fig. 4b).

Proof. $\Gamma(r_0)C_M(r_0)$ is a constant which depends only on Γ, but not on r_{12}. Similarly, if we restrict ourselves to distances $r_{12} > r_0'$, where r_0' is any positive number, the lower bound (31) holds true with $C_m(r_{12})$ replaced by the positive constant $C_m(r_0')$. So, this constant depends only on Γ, too. Finally, if $\Gamma(r_{12})$ is not infinite everywhere, we can choose r_0 such that the upper bound is finite. To prove the converse statement, it is sufficient to note that a 2-graph that decreases like $\exp(-nr)$ must satisfy $\lambda = n$, and thus has the structure given in Fig. 4b, by the max-flow, min-cut theorem.

We can see here that λ is an *integer*, contrary to what had been previously stated.[11]

Corollary 5.2. The irreducible 2-prototypes dominant at infinity decay exponentially like e^{-r} and have at least one isthmus (see Fig. 4a).

Proof. Irreducible 2-graphs satisfy the condition $\lambda \geqslant 1$. Moreover, the class of 2-prototypes decaying exponentially like e^{-r} is not void, because it contains the class of 2-graphs made of alternating lines e^{-r}/r and $(e^{-r}/r)^2$, considered by Mitchell, Ninham, and others.[(1)] Finally, a 2-graph decaying exponentially like e^{-r} has an isthmus, by Corollary 5.1.

In a subsequent paper we will make use of these corollaries to find the exact asymptotic behavior of the 2-prototypes that are dominant at large distances.[(19)] Now, we turn to the proof of Theorem 5.1.

[11] In Refs. 7 and 8, Deutsch *et al.* also made use of a second method to find the asymptotic behavior of 2-graphs, different from the one described in footnote 3. They applied it to the simplest 2-graphs of order three and four in the plasma parameter, and found an asymptotic behavior in e^{-Br}/r for these 2-graphs. But the B's that they obtain are noninteger, real numbers, which is impossible, by our Theorem 5.1. Their error comes from the fact that they Fourier transform $\Gamma(r)$, and make use of the equivalence $1 - a^2k^2 \sim (1 + a^2k^2)^{-1}$. This relation is exact for small values of k, but introduces poles in the complex plane which do not belong to the Fourier transform of $\Gamma(r)$. This therefore gives an incorrect decay. The same type of reasoning is used in the Ornstein–Zernike theory for the correlation function $h(r)$. Therefore, the exponential decay that they obtain for $h(r)$ could be incorrect, too, for the same reason.

5.1. Upper Bound for $\Gamma(r)$

To obtain an upper bound for $\Gamma(r)$ that decays exponentially like $e^{-\lambda r}$, we will exploit the inequality $\lambda(1, 2) \leqslant h_\Gamma$. But if we try to introduce it directly into (12), we run into difficulties. These come from the infinite volume of integration, because the function $1/r$ is not integrable in it. To get around this difficulty, we take a convergence factor $e^{-r_0 R}$ with each nonintegrable factor $1/R$, where r_0 is an arbitrary, positive number. In other terms, we rewrite (12) in the form

$$\Gamma(r_{12}) = r_{12}^{3k-l} \int_{\Lambda_\infty{}^k} \exp[-(r_{12} - r_0)h_\Gamma] \prod_{L \in \mathscr{L}\Gamma} \frac{\exp(-r_0|\mathbf{R}_L|)}{|\mathbf{R}_L|} d\mathbf{R}_3 \cdots d\mathbf{R}_{k+2} \tag{35}$$

Then, provided $r_{12} \geqslant r_0$, the theorem of the means [20] gives

$$\Gamma(r_{12}) \leqslant r_{12}^{3k-l} e^{-\lambda r_{12}} r_0^{l-3k} e^{\lambda r_0} \Gamma(r_0), \qquad \forall r_{12} > r_0 \tag{36}$$

This proves the rhs inequality in (31), with $C_M(r_0) = r_0^{l-3k} e^{\lambda r_0}$. To conclude the proof, we can choose r_0 such that $\Gamma(r_0)$ is finite. This is possible, because we assumed at the beginning that $\Gamma(r)$ is not infinite everywhere. Therefore, the upper bound (31) is finite, too.

5.2. Lower Bound for $\Gamma(r)$

To obtain a lower bound for $\Gamma(r)$ that decreases exponentially like $e^{-\lambda r}$, we will take advantage of inequality (29). We are thus led to restrict the domain of integration to the domain of validity of this inequality. This gives actually a lower bound on $\Gamma(r)$ because its integrand is positive. We have

$$\Gamma(r_{12}) \geqslant r_{12}^{3k-l} \frac{\exp[-\lambda r_{12}(1 + 2\eta)]}{(1 + 2\eta)^\lambda} C_1(\eta, r_{12}) C_2(\eta, r_{12}) \tag{37}$$

$C_1(\eta, r_{12})$ stands for the contribution of the points $\mathbf{R}_{i_1}, \mathbf{R}_{j_1}, \ldots$ with domain restricted to the sphere $S_1(\eta)$:

$$C_1(\eta, r_{12}) = \int_{[S_1(\eta)]^{k_1}} \prod_{L \in \mathscr{L}\Gamma_1} \frac{\exp(-r_{12}|\mathbf{R}_L|)}{|\mathbf{R}_L|} d\mathbf{R}_{i_1} \cdots d\mathbf{R}_{j_1} \tag{38}$$

Similarly, $C_2(\eta, r_{12})$ is defined by replacing, in (38), $S_1(\eta)$ and $\mathscr{L}\Gamma_1$ by $S_2(\eta)$ and $\mathscr{L}\Gamma_2$.

For the lower bound (37) to have the required behavior at infinity, we can choose $\eta = r_{12}^{-1}$. Then, by homogeneity considerations, we obtain

$$C_1(r_{12}^{-1}, r_{12}) = r_{12}^{l_1 - 3k_1} C_1(1, 1) \tag{39}$$

where l_1 and k_1 stand, respectively, for the number of lines and field points of Γ_1. By putting this relation into (37), and making use of the relations $k = k_1 + k_2$ and $l = l_1 + l_2 + \lambda$, we obtain the following lower bound, valid for any r_{12}:

$$\Gamma(r_{12}) \geqslant C_m(r_{12}) r_{12}^{-\lambda} e^{-\lambda r_{12}} \tag{40}$$

with $C_m(r_{12})$ defined by (34). This completes the proof of Theorem 5.1.

5.3. An Improved Lower Bound

We can improve (32) by making use of all possible line-cutsets between the root points, provided we restrict ourselves to distances $r_{12} > 2$.

Theorem 5.2. We have the lower bound

$$\sum_{\mathscr{C}} C_m(\mathscr{C}) r_{12}^{-\nu} e^{-\nu r_{12}} \leqslant \Gamma(r_{12}), \qquad r_{12} > 2 \tag{41}$$

with

$$C_m(\mathscr{C}) = C_1(\mathscr{C}) C_2(\mathscr{C}) e^{-2\nu} 2^{-\nu} \tag{42}$$

where the sum in (41) runs over all possible line-cutsets $\mathscr{C}$ of Γ, and ν is the number of lines in $\mathscr{C}$. Here $C_1(\mathscr{C})$ denotes the right-hand side of (38), where Γ_1 is the connected part of $\Gamma - \mathscr{C}$ that contains the root point 1.

Proof. Let us consider any given line-cutset $\mathscr{C}$ with ν lines ($\nu \geqslant \lambda$). It induces a partition of the field points into two sets $\mathscr{P}\Gamma_1(\mathscr{C})$ and $\mathscr{P}\Gamma_2(\mathscr{C})$, and a subset $\sigma(\mathscr{C})$ of $\Lambda_\infty{}^k$, by the procedure described in Section 4.2. We define $\sigma(\mathscr{C})$ to be

$$\sigma(\mathscr{C}) = \{(\mathbf{R}_3,\ldots,\mathbf{R}_{k+2}) \mid i \in \mathscr{P}\Gamma_1(\mathscr{C}) \Rightarrow \mathbf{R}_i \in S_1, \quad i \in \mathscr{P}\Gamma_2(\mathscr{C}) \Rightarrow \mathbf{R}_i \in S_2\}$$

But, for $r_{12} > 2$, we have $\sigma(\mathscr{C}_i) \cap \sigma(\mathscr{C}_j) = \varnothing$ for two different line-cutsets $\mathscr{C}_i$ and $\mathscr{C}_j$, because $S_1(r_{12}^{-1})$ and $S_2(r_{12}^{-1})$ are disjoint. This implies that the integral in $\Lambda_\infty{}^k$ is larger than the sum of the integrals in the subdomains $\sigma(\mathscr{C}_i)$. Therefore we obtain (40) by the same reasoning that was used in the preceding subsection.

6. DIVERGENT 2-PROTOTYPES

From now on, we will make use of our conventions to represent 2-graphs.

It is well known that a 2-graph with lines $f_L(r) = (e^{-r}/r)^{k_L}$ is divergent if one of the f_L is nonintegrable, i.e., if $k_L \geqslant 3$. It is less obvious that 2-graphs with integrable lines ($k_L < 3$ for any L) can also be divergent, but this is nevertheless true.[12] We will show this as an application of Theorem 5.1.

[12] We exhibit here divergent 2-graphs that occur in the Abe–Meeron theory of ionized systems. Cohen and Murphy have also met divergences[1] when computing certain 2-graphs that occur in this theory. But the 2-graphs that they compute are *convergent*, as one can see by applying the method we developed elsewhere,[9,10] and which gives upper bounds for general n-graphs. The divergences that they observe mean that their method of computation, which makes use of a development of $B(k)$ [the Fourier transform of $B(r)$] in a double series of ϵ and $\ln \epsilon$, is not valid for high orders in ϵ and $\ln \epsilon$. This is probably because developing $B(k)$ in powers of ϵ and $\ln \epsilon$ amounts more or less to developing $B(r)$ in powers of ϵ, which gives back, for high orders in ϵ, the nonintegrable lines $(e^{-r}/r)^n$, $n \geqslant 3$.

This implies that the Abe–Meeron[3] development of $w(r)$ contains divergent 2-graphs, although all the divergences coming from nonintegrable lines have been removed. This implies also that further resummations would be needed to take care of these divergent 2-graphs. But we will not study this problem here.

Let us then consider 2-graphs with lines $f_L(r) = e^{-r}/r$ (i.e., $k_L = 1$ for any L). Such 2-graphs belong to the Abe–Meeron development of $w(r)$ if these are, moreover, simple, irreducible 2-prototypes.[2,3] We are now going to prove:

Theorem 6.1. A sufficient condition for a 2-graph with Debye–Hückel lines to be divergent is that

$$l > 3k + \lambda \tag{43}$$

For any $k > 5$, there are simple, irreducible 2-prototypes satisfying this condition.

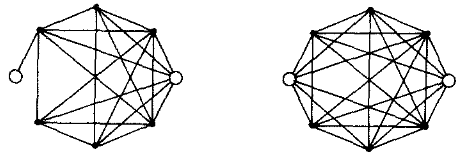
Proof. The lower bound (32) cannot decay less rapidly at large distances than the upper bound (31). This implies that

$$l - 3k \leqslant \lambda \tag{44}$$

If this condition is not satisfied, the inequality (31) can hold only if $\Gamma(r_0)$ is infinite. But r_0 is arbitrary, and then $\Gamma(r_{12})$ is infinite for any r_{12}. In other words, a sufficient condition for $\Gamma(r_{12})$ to be divergent is that (43) holds. To construct such divergent 2-prototypes for any k, let us note that, according to (43), they must have a sufficiently large number of lines. We can consider, for example, a 2-graph with k field points, and the maximum number of lines, that is,

$$l = \tfrac{1}{2}(k + 2)(k + 1) - 1 \tag{45}$$

This is clearly a simple, irreducible 2-prototype, because each field point has a degree larger than two for $k \geqslant 2$. [And then the associated 2-graph obtained by taking $f_L(r) = e^{-r}/r$ for any L actually occurs in the Abe–Meeron development of $w(r)$, as we already noticed.] Moreover, we have also



(a) (b)

Fig. 7. Two examples of divergent 2-graphs. (a) $l = 20, k = 6, \lambda = 1$. (b) $l = 25, k = 6, \lambda = 6$.

$\lambda = k$. By the relation (44), we thus see that $\Gamma(r)$ is divergent whenever $k > 5$, although all its lines are equal to e^{-r}/r and therefore are integrable. We give some examples of divergent 2-prototypes in Fig. 7.

The lower bound (32) enables us also to conclude that a given 2-graph is divergent if it contains subgraphs that are known to be divergent.

Theorem 6.2. If there is a cutset $\mathscr{C}$ such that one of the components of $\Gamma - \mathscr{C}$ is divergent, then $\Gamma(r_{12})$ is divergent.

Proof. The idea consists in finding a lower bound for $C_1(1, 1)$ by means of $\Gamma_1(\mathbf{r}_1)$. From (39), we have

$$C_1(\alpha, \alpha^{-1}) = \alpha^{l_1 - 3k_1} C_1(1, 1) \tag{46}$$

for any α. Moreover, if $\alpha > 1$, we have also

$$C_1(\alpha, 1) < C_1(\alpha, \alpha^{-1}) \tag{47}$$

because of the inequality $\exp(-R) \leqslant \exp(-\alpha^{-1}R)$. But, for increasing values of α, $C_1(\alpha, 1)$ increases monotonically to $\Gamma_1(\mathbf{r}_1)$, if this quantity exists (by the Lebesgue dominated convergence theorem[14]). So, to a given $\epsilon > 0$, we can associate an $\alpha_1 > 0$ such that

$$C_1(\alpha, 1) > (1 - \epsilon)\Gamma_1(\mathbf{r}_1), \qquad \forall \alpha > \alpha_1 \tag{48}$$

By combining (34) with (46)–(48), we find

$$C_m(r_{12}) \geqslant \Gamma_1(\mathbf{r}_1)\Gamma_2(\mathbf{r}_2)C_m'(r_{12}) \tag{49}$$

with

$$C_m'(r_{12}) = \alpha_1^{l_1 - 3k_1}\alpha_2^{l_2 - 3k_2}(1 - \epsilon)^2 e^{-2\lambda}(1 + 2r_{12}^{-1})^{-\lambda} \tag{50}$$

This proves Theorem 6.2 for a given minimal line-cutset $\mathscr{C}$. The proof for any line-cutset is a consequence of our improved lower bound (41).

In this section, we have proved that there are irreducible 2-graphs with Debye-Hückel lines that are infinite everywhere. This brings us to make some remarks.

Remark 1. The divergences that we found were detected by looking at the behavior of 2-graphs at large distances, but it is clear that these divergences come from the short distance behavior of the Debye–Hückel function, as was already mentioned in footnote 8.

Remark 2. Our upper bound (31) solves completely the problem of finding upper bounds finite at large distances, for any 2-graph that is not infinite everywhere. A distinct but important problem is to know whether a given 2-graph is finite or infinite. This is a very difficult problem, which we were able to avoid, thanks to the factor $\Gamma(r_0)$ in (31). The results of this section give yet a partial answer to this problem, by displaying an infinite class of irreducible 2-graphs that are infinite everywhere. But these results are by no means able to give a complete solution to this problem. In particular, these cannot enable us to find the whole set of infinite 2-graphs, nor to find any finite one. Completely different techniques must be developed to prove that a given 2-graph is finite. We have developed such a technique elsewhere.[9,10] By means of this technique, we have been able, for example, to prove that a 2-graph $\Gamma(r)$ with l lines and k field points is finite if $l < 3k$ $\{$it is bounded by $\left[\int (e^{-r}/r)^{l/k}\, dr\right]^k < +\infty\}$, provided Γ can be uniformly covered[13] by a set of spanning two-rooted trees.[14] This enabled us to find an infinite class of 2-graphs that are finite.

Remark 3. The results of this section show that the "bounds" of Deutsch *et al.* can happen to be infinite even if $\Gamma(r)$ is finite, because these are sums of 2-graphs that are *different* from $\Gamma(r)$. On the contrary, our bound (31) is finite whenever $\Gamma(r)$ is not infinite everywhere, because this bound makes use of $\Gamma(r)$ itself.

7. GENERALIZATION TO 2-GRAPHS WITH NONINTEGER POWERS OF DEBYE–HÜCKEL LINES

Let us first rewrite $\Gamma(r_{12})$ when Γ is represented according to our convention:

$$\Gamma(r_{12}) = \int_{\Lambda_\infty{}^k} e^{-r_{12}h_\Gamma} \prod_{L\in\mathscr{L}\Gamma} \left(\frac{1}{|\mathbf{R}_L|}\right)^{k_L} d\mathbf{R}_3 \cdots d\mathbf{R}_{k+2} \tag{51}$$

h_Γ is now defined by the identity

$$h_\Gamma(\mathbf{R}_3,\ldots,\mathbf{R}_{k+2}) = \sum_{L\in\mathscr{L}\Gamma} k_L|\mathbf{R}_L| \tag{52}$$

In the preceding sections, the k_L were all *integers*. We showed that the exponential decay of $\Gamma(r_{12})$ is determined by the minimum of h_Γ, which is equal to the local line connectivity $\lambda(1, 2)$ of Γ. We now want to extend this result to the case where the k_L can take *real* values. To this end, we prove that the minimum μ of h_Γ is equal to a straightforward generalization of $\lambda(1, 2)$, namely the maximal flow[16] between the root points 1 and 2, in a network $N\Gamma$ associated with Γ.

[13] A graph is covered by a set of subgraphs if their union is equal to the graph itself. A covering is uniform if each line of the graph belongs to the same number of subgraphs.

[14] A spanning two-rooted tree of a graph Γ is a subgraph that contains all the points of Γ and that is made of two disjoint trees, each one containing one root point.

In fact, we could even avoid introducing the notion of flows in networks by reducing the computation of μ for real k_L to the computation of the minimum of a new function H_Γ with integer k_L. To this end, we have just to approximate each k_L as closely as we want by a fraction, and then reduce all of them to the same denominator. We obtain in this way an approximation to μ as precise as we want. Nevertheless, we have chosen to make use of flows because they arise quite naturally, without introducing any auxiliary function or limiting process. This enables us also to use known algorithms to compute μ (Ref. 16, pp. 17–18). On the other hand, this brings some extra complications in the notation, and forces us to give some new definitions that we will need later.

Definition 7.1. A directed network is a set of points $i, j, \ldots$ together with a set of *ordered* pairs (i, j) of points, referred to as arcs.

To each graph Γ, we can now associate a network $N\Gamma$ in the usual way (Ref. 16, p. 23): each (undirected) line of Γ gives rise in $N\Gamma$ to a pair of oppositely directed arcs, each having capacity equal to the old line. An arc $A = (i, j)$ is depicted by a line joining i and j, and an arrowhead oriented toward j (Fig. 8). The set of arcs of $N\Gamma$ is denoted by $\mathscr{A}\Gamma$.

Definition 7.2. A flow of value v from the root points 1 to 2 in $N\Gamma$ is a function f from $\mathscr{A}\Gamma$ to nonnegative reals that satisfies the linear equations and inequalities (Ref. 16, p. 4)

$$\sum_{A\in\omega^+(i)} f(A) - \sum_{A\in\omega^-(i)} f(A) = \begin{cases} v & i = 1 \\ 0 & i \neq 1, 2 \\ -v & i = 2 \end{cases} \tag{53}$$

$$0 \leqslant f(A) \leqslant k_A \qquad \text{for all} \quad A \in \mathscr{A}\Gamma \tag{54}$$

where $\omega^+(i)$ denotes the set of all arcs issuing from point i:

$$\omega^+(i) = \{A \in \mathscr{A}\Gamma \mid A = (i, j), \quad j \in \mathscr{P}\Gamma\} \tag{55}$$

and similarly, $\omega^-(i)$ denotes the set of all arcs going to i:

$$\omega^-(i) = \{A \in \mathscr{A}\Gamma \mid A = (j, i), \quad j \in \mathscr{P}\Gamma\} \tag{56}$$

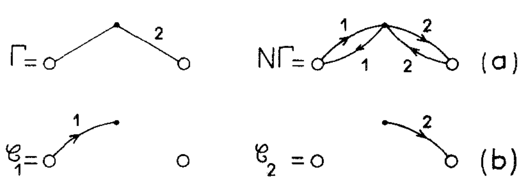


Fig. 8. (a) Construction of the network $N\Gamma$ associated to the 2-rooted graph Γ. (b) Two examples of arc-cutsets. Only the arc-cutset C_1 is minimal.

The definition of a line-cutset of a graph extends straightforwardly to a network:

Definition 7.3. An arc-cutset (or cut) in $N\Gamma$ separating the root points denotes the set of all arcs that lead from X to a point not belonging to X, where X is a set of points containing the root point 1, but not 2.

Definition 7.4. The capacity of an arc-cutset $\mathscr{C}$ is the sum of the capacities of all the arcs of $\mathscr{C}$.

We will denote by $\lambda(1, 2)$ the minimal capacity of all arc-cutsets separating the root points.

7.1. Computation of the Minimum μ of h_Γ

From the definition of an arc-cutset and the construction of Section 4.2, we see immediately that μ is smaller than or equal to the minimal capacity $\lambda(1, 2)$:

$$\mu \leqslant \lambda(1, 2) \tag{57}$$

To prove the converse inequality, we first note that the minimal capacity $\lambda(1, 2)$ is equal to the maximal flow from 1 to 2 in $N\Gamma$, by the max-flow, min-cut theorem. Then, we show that μ is larger than or equal to the maximal flow $\lambda(1, 2)$. To this end, we will express flows in arc-chain form (Ref. 16, p. 7).

Let f be a maximal flow of value $\lambda(1, 2)$, expressed in the usual node-arc form. We will require that f vanishes on one of the two oppositely directed arcs associated to the line L, for each L. We know that this is possible (Ref. 16, p. 23). We know also (Ref. 16, Lemma 2.1) that if f is a node-arc flow from 1 to 2 having positive value $v(f)$, then there is an oriented chain from 1 to 2 such that $f > 0$ on all arcs of the chains. Let us call this chain C_1, and $h(C_1)$ the maximal admissible flow in C_1 (Ref. 16, p. 7):

$$h(C_1) = \min_{A \in C_1} f(A) \tag{58}$$

We have $h(C_1) \leqslant v(f)$, because $v(f)$ is maximal. Moreover, the quantity $f_1(A)$, defined by

$$f_1(A) = f(A) - a_{A1} h(C_1) \tag{59}$$

(where a_{A1} is equal to 1 if $A \in C_1$ and is zero otherwise), is a flow (Ref. 16, p. 7). If $h(C_1) = v(f)$, we have found a flow in arc-chain form which is equal to $\lambda(1, 2)$. If $h(C_1) < v(f)$, the flow f_1 has positive value and we can iterate the preceding algorithm. As there is a finite number of chains from 1 to 2, we will end up in a finite number of steps with a chain C_{m+1} satisfying the equality $h(C_{m+1}) = v(f_m)$, where f_m is the flow obtained after the mth iteration. In other terms, we have found $m + 1$ oriented chains going from 1 to 2, such that

$$\sum_{i=1}^{m+1} h(C_i) = \lambda(1, 2) \tag{60}$$

$$f(A) = \sum_{i=1}^{m+1} a_{Ai} h(C_i) \tag{61}$$

where a_{Ai} is equal to 1 if $A \in C_i$, and is zero otherwise.

Let us now write

$$\mathbf{R}_{12} = \sum_{A \in \mathscr{L} C_i} \mathbf{R}_A = \sum_{A \in \mathscr{A}\Gamma} a_{Ai} \mathbf{R}_A \tag{62}$$

where $\mathbf{R}_A$ denotes the quantity $\mathbf{R}_i - \mathbf{R}_j$, i and j being the end points of arc A. By applying the triangular inequality to (62), summing over i, and making use of (61), we get

$$\lambda(1, 2) \leqslant \sum_{A \in \mathscr{A}\Gamma} f(A)|\mathbf{R}_A| \tag{63}$$

But the flow satisfies the constraints (54) and, moreover, to each line L is associated at most one arc with a strictly positive flow $f(A)$. This implies that the rhs of (63) is majorized by the quantity $\sum_{L \in \mathscr{L}\Gamma} k_L |\mathbf{R}_L| = h_\Gamma$ and so we obtain the desired inequality:

$$\lambda(1, 2) \leqslant \mu \tag{64}$$

By combining this inequality with (57), we get:

Lemma 7.1. The minimum μ of h_Γ is equal to the maximal flow $\lambda(1, 2)$ in $N\Gamma$.

7.2. Upper and Lower Bounds for $\Gamma(r)$

Theorem 5.1 extends immediately into the following:

Theorem 7.1. A given irreducible 2-graph $\Gamma(r_{12})$ with lines $f_L(r) = (e^{-r}/r)^{k_L}$ and k field points has the same upper and lower bounds as in Theorem 5.1, provided the definitions of l, λ, and $C_1(1, 1)$ are modified in the following way:

(i) l now stands for the total capacity of Γ: $l = \sum_{L \in \mathscr{L}\Gamma} k_L$.

(ii) λ stands for the maximal flow in the network $N\Gamma$ obtained from Γ by replacing each line with a pair of oppositely directed arcs, each having capacity equal to the old line.

(iii) $C_1(1, 1)$ is obtained by replacing $|\mathbf{R}_L|^{-1}$ by $|\mathbf{R}_L|^{-k_L}$ in (38).

Finally, the sufficient condition (43) for a 2-graph with lines $f_L(r) = (e^{-r}/r)^{k_L}$ remains unchanged, with these definitions of l and λ.

8. DISCUSSION AND COMMENTS

We have shown the fundamental importance of the local line connectivity λ for the asymptotic behavior of 2-graphs in the case of the one-component plasma. This is in fact fairly general. For example, we have proved also[6] that 2-graphs with lines $f_L(r) \sim r^{-k_L}$, which occur in the theory of neutral systems, decay like $r^{-\lambda}$.

But we would like to emphasize here that it is in the case of the OCP that the local line connectivity arises in the most simple and natural manner. To illustrate this point, we need only to remark that it takes just two lines to prove that a 2-graph decays exponentially at least as fast as e^{-r}. The first line consists in writing Eq. (35), which reexpresses $\Gamma(r_{12})$ in a form most convenient for our purpose, by displaying the quantity h_Γ and the absolutely integrable factor $\prod [\exp(-r_0|\mathbf{R}_L|)]/|\mathbf{R}_L|$. The second line reads

$$1 \leqslant \sum_{L \in \mathscr{L}\Gamma} |\mathbf{R}_L| \leqslant h_\Gamma,$$

which is an immediate consequence of the existence, in an irreducible 2-rooted graph, of at least one chain linking the root points. Finally, to prove that a 2-graph decays exponentially at least as fast as $e^{-\lambda r}$, we need only, in a third step, to extract from h_Γ as many quantities $\sum_{L \in \mathscr{L}C} |\mathbf{R}_L|$ as possible.

Such simplicity no longer exists in the case of neutral systems, and the proof is much more opaque. This is mainly because one cannot isolate any absolutely integrable factor from the integrand of $\Gamma(r_{12})$ without weakening the decay of the upper bound.

On the other hand, in the case of the OCP, we have been able to obtain only upper and lower bounds because the minimum of h_Γ is not reached, in general, at a point, but at a set of points. For neutral systems, on the contrary, it is possible to go further and obtain the exact asymptotic behavior of $\Gamma(r_{12})$, because the field points give a nonnegligible contribution only when they are in the vicinity of the root points.

Finally, we mention that the results of this paper have been applied elsewhere[19] to justify the corrections (1) to the Debye length.

ACKNOWLEDGMENTS

I am very grateful to Prof. J. Chapelle for constant support and encouragement during this work, and to Prof. J. Lebowitz for extremely useful comments and advice. I am also very grateful to Prof. P. Choquard,

C. Deutsch, H. Friedman, J. Groeneveld, H. Kunz, D. Ruelle, B. Souillard and G. Stell for very useful discussions and correspondence.

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